Malaysian Journal of Mathematical Sciences 19(4): 1397–1415 (2025)
https://doi.org/10.47836/mjms.19.4.12

Malaysian Journal of Mathematical Sciences

Journal homepage: https://mjms.upm.edu.my

Metric Dimension of 4-Regular Cyclic Bipartite Graphs

Bharani Dharan, K.[1] and Radha, S.*[1]

[1]*Department of Mathematics, School of Advanced Sciences, Vellore Institute of Technology, 600127 Chennai, India*

E-mail: radha.s@vit.ac.in
**Corresponding author*

Received: 9 February 2024
Accepted: 28 May 2025

Abstract

The metric dimension of a graph, denoted by $\dim(G)$, represents the minimum number of landmarks required to uniquely identify all vertices based on their distances to these landmarks. This parameter plays a crucial role in interconnection networks, facilitating applications such as efficient routing and system verification. Cyclic bipartite graphs, which model systems with periodic structures like communication rings and circular sensor networks, present particular challenges due to their inherent symmetry. Determining the metric dimension in such graphs is essential for efficient vertex identification and resource optimization. In this paper, we compute the metric dimension of 4-regular cyclic bipartite graphs, denoted by $CB_{4,n}$. Specifically, for all $n \geq 5$, we establish that $\dim(CB_{4,n}) = 4$ when $n \not\equiv 1 \pmod 3$, and $\dim(CB_{4,n}) = 5$ when $n \equiv 1 \pmod 3$, thus providing a precise characterization based on graph parameters. These findings not only advance the theoretical understanding of cyclic bipartite structures, but also offer practical implications for the design of robust network topologies, optimal sensor placement in cyclic environments, and efficient localization in robotics and surveillance systems. By minimizing the number of required landmarks, our results contribute significantly to the optimization of resource allocation in structured and repetitive networks.

Keywords: cyclic bipartite graphs; resolving set; metric basis; metric dimension.

1 Introduction

The concept of metric dimension in graph theory finds its roots in the practical challenge of locating a point in a network by its distances to a selected group of reference points [24]. One of its earliest motivations was the problem of locating satellites in space based on signals received from a few ground stations [56]. This principle has found modern applications in areas such as robotic navigation, where a robot must determine its location within a map from distance measurements to certain landmarks. In communication networks, the problem arises when determining the position of a faulty node based on its distance from monitoring devices. Similarly, in GPS technology, satellites measure distances to known fixed stations, essentially solving a metric dimension problem. This wide array of applications makes the study of metric dimension a cornerstone in discrete mathematics and theoretical computer science.

Let $G = (V(G), E(G))$ be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. The distance $d(u, v)$ between any two vertices $u, v \in V(G)$ is the length of the shortest path connecting them. For an ordered set $\mathfrak{R} = \{v_1, v_2, \ldots, v_k\}$ of vertices in G, the metric representation of a vertex $u \in V(G)$ with respect to $\mathfrak{R}$ is defined as:

$$r(u|\mathfrak{R}) = (d(u, v_1), d(u, v_2), \ldots, d(u, v_k)).$$

If every vertex in G has a unique metric representation with respect to $\mathfrak{R}$, then $\mathfrak{R}$ is called a resolving set. The smallest cardinality among all resolving sets of G is called the metric dimension of G, denoted by $\dim(G)$.

The concept of resolving sets and metric dimension was independently introduced by Slater [54] and studied by Harary and Melter [22]. Subsequent foundational work was carried out by Chartrand et al. [17] and Johnson [28], who demonstrated its significance in chemistry, particularly in distinguishing structural isomers using chemical graph theory. Studies on the metric dimension of polycyclic aromatic hydrocarbons [5], the honeycomb rhombic torus [14], and carbon nanotube Y-junctions [40] highlight the significance of metric dimension in chemical graph theory. The computational complexity of determining the metric dimension was studied by Khuller et al. [30], who showed that finding a minimum resolving set is NP-hard. Manuel et al. [37] extended these results to bipartite graphs, proving the problem remains NP-complete in this setting. Similar results related to computational complexity were also shown for directed graphs by Ramesh et al. [47].

Given its computational difficulty, much of the work has focused on determining the metric dimension for special classes of graphs. For example, Behtoei and Erfanian [11] applied the concept to network verification, while Sebö and Tannier [51] used the metric dimension in the problems like combinatorial search and information theory. It has been employed in image processing and pattern recognition tasks, including coin weighing problems [55] and geometric routing protocols [33, 39].

Extensive studies have been conducted on the metric dimension of standard graph families such as trees, 2-trees [12], Grassmann graphs [9], wheel related graphs [53], multi-dimensional grids [30], and generalized fat trees [44]. The metric dimension of torus networks was investigated in [36], while honeycomb networks were studied through the duality approach in [35] and under specific conditions in [50]. Petersen graphs were studied by [52], and advanced interconnection topologies like star fan graph [43], Beneš networks [37], hypercubes [10], 6-regular circulant graphs [25], 8-regular circulant graphs [19], t-regular circulant graphs [57], four dimensional Klein bottle [31] and Illiac networks [49] have also been explored. Other graph classes studied include enhanced hypercubes [48], Cayley graphs [18], unicyclic graphs [42], Kneser and Johnson graphs [8], Cartesian product graphs [16], and fullerene graphs [1]. Work on infinite graphs [15], convex polytopes [32], permutation graphs [21], incidence graphs [7], and wheel-related graphs [53] has further diversified the scope.

In parallel and distributed computing, reliable interconnection network design is essential. The fault-tolerant metric dimension of such networks was studied in [2], where the authors investigated fractal cubic networks for their robustness. Work by Prabhu et al. [45] refined the understanding of silicate and butterfly networks. Irregular convex triangular networks were analyzed in [46], offering new avenues for scalable fault-tolerant network topologies.

The study of the metric dimension in bipartite graphs is particularly important due to their relevance in modeling real-world systems [6]. Hasni and Peng [23] have identified the chromatically unique bipartite graphs. Then, Mahanani and Ismiarti [34] have classified some bipartite graphs associated to elements and class equivalences of a finite heap. Applications range from representing molecular structures in chemistry to construct block designs in experimental statistics [29], modeling biological data in healthcare [59], and optimizing big data and cloud computing platforms [58]. In coding theory, bipartite graphs form the underlying structure for many error-correcting codes, and in computer science, they are widely used in recommendation systems, ranking algorithms, advertisement targeting models and data structures. Recent research has increasingly focused on computing metric-related parameters for finite graphs representing drug molecules, including those used for fosmidomycin-based antimalarial treatments [26], COVID-19 antivirals [38], and antidepressants [41]. Additionally, variants such as the locating number [4] and partition dimension [3] have been studied for anticancer drug structures.

When compared with those efforts, the results presented in this work offer a more general and foundational breakthrough in graph theory as the order of 4-regular cyclic bipartite graphs depends on n, particularly by bridging algebraic properties with combinatorial metric constraints. The basic definitions of cyclic bipartite graphs and their applications are presented in Section 2. The main results concerning the metric dimension, along with their proofs, are provided in Section 3. Finally, the conclusion and potential directions for future work are discussed in Section 4.

2 Cyclic Bipartite Graphs

Cyclic bipartite graphs, introduced by Jiang and Lu [27] are particularly notable for their utility in network communication and error correction, addressing challenges in reliable data exchange [13, 20]. Let $x \in [a, b]$ be the set of all integers, where $a, b \in \mathbb{Z}$. Hereafter, all the operations are carried with modulo n (if $i = 2$ and $j = n-2$, then $i+j = 0$). Let $CB_{\Delta,n}$ denote the cyclic bipartite graph, where Δ and n are two positive integers satisfying $2 \leq \Delta \leq n$. The vertex set $V(CB_{\Delta,n})$ comprises two disjoint sets $X^a = \{x^a_j : 0 \leq j \leq n-1\}$ and $X^b = \{x^b_j : 0 \leq j \leq n-1\}$. The edge set $E(CB_{\Delta,n}) = \{(x^a_j, x^b_{j+j'}) : 0 \leq j \leq n-1, 0 \leq j' \leq \Delta - 1\}$. An example for cyclic bipartite graph $(CB_{4,8})$ is given in Figure 1. For $n = 4$, $CB_{4,n}$ is a complete bipartite graph $K_{4,4}$ whose metric dimension is 6. Hereafter, let us assume that $n \geq 5$.

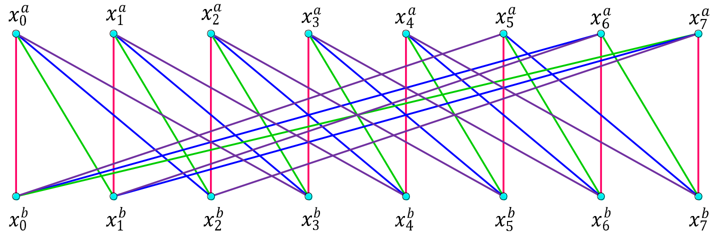

Figure 1: Cyclic bipartite graph ($CB_{4,8}$).

Cayley graphs, especially those derived from dihedral groups, offer a rich framework for exploration in algebraic graph theory. Owing to their symmetry and high regularity, these graphs serve as attractive candidates for studying structural graph parameters such as the metric dimension. Moreover, Cayley graphs are not merely of theoretical interest; they play a crucial role in modeling interconnection networks in parallel computing and communication systems [18]. A particularly interesting subclass in this context is the family of 4-regular cyclic bipartite graphs, which are both bipartite and Cayley graphs over dihedral groups. Specifically, these graphs are isomorphic to Cayley graphs over dihedral groups of order $2n$, with generating set $\{y, xy, x^2y, x^3y \,:\, x^n = y^2 = e\}$.

Studying the metric dimension of $CB_{4,n}$ thus provides a twofold contribution: it advances the structural theory of bipartite graphs and offers new insights into metric parameters on algebraically constructed Cayley graphs. While considerable attention has been paid in the literature in computing the metric dimension of specific families of graphs such as trees, hypercubes, grids, and circulant graphs as given in Section 1. These structures often have regularities that make them easier to analyze. In contrast, while families like trees and grids have been thoroughly investigated, cyclic bipartite graphs represent a compelling next step in exploring the interaction between symmetry, regularity, and resolvability. Their cyclic automorphisms and bipartite structure yield new resolving set behaviors that are not trivially inferred from known results. Moreover, the $CB_{4,n}$ graphs, being isomorphic to Cayley graphs over dihedral groups, offer a natural algebraic setting to study metric dimension under group-theoretic symmetry.

In this paper, we address this gap by determining the exact metric dimension of the cyclic bipartite graph $CB_{4,n}$ for all $n \geq 4$. This contribution represents a meaningful advancement in understanding how the metric dimension behaves under both cyclic and bipartite constraints. Our results not only build on the foundational theories of graph metric parameters but also have potential applications in fields such as network design, error correction, and modeling of complex systems.

3 Main Results

The motivation to investigate the metric dimension of $CB_{4,n}$ stems from both structural richness and algebraic significance. The regularity and symmetry of these graphs make them ideal for testing the boundaries of metric dimension theory. Additionally, as Cayley graphs over dihedral groups, they encapsulate both geometric cyclicity and reflectional symmetry, traits that critically influence the uniqueness of metric representations. This dual nature positions $CB_{4,n}$ as a natural and nontrivial candidate for metric dimension analysis, rather than an arbitrary choice.

The lower bounds for the metric dimension of cyclic bipartite graphs for $n \geq 5$ are given in Lemma 3.1 and Lemma 3.2. Then, the exact value of the metric dimension of $CB_{4,n}$ is presented in Theorems 3.1 and 3.2.

Lemma 3.1. *For any* $n \geq 5$, $\dim(CB_{4,n}) \geq 4$.

Proof. Let W be any resolving set of $CB_{4,n}$ having cardinality less than four. Without loss of generality, let us assume that $|W| = 3$, $x_0^a \in W$. Let us take the subset U of consecutive vertices such that $|D_{x_0^a}(U)| = 1$ with cardinalities 4 or 5 or 6 according as $n \equiv 2 \pmod 3$ or $n \equiv 0 \pmod 3$ or $n \equiv 1 \pmod 3$ respectively. Then, all the vertices in U need at least three more vertices from W to get resolved, which is a contradiction. □

Due to the symmetric property of $CB_{4,n}$, the above proof will hold good for any vertex $v \in V(CB_{4,n})$.

Lemma 3.2. *For any* $n \geq 7$ *and* $n \equiv 1 \pmod 3$, $\dim(CB_{4,n}) \geq 5$.

Proof. Let us consider $CB_{4,n}$ with $n \equiv 1 \pmod 3$. Without loss of generality, let us assume that W be the resolving set of $CB_{4,n}$ with cardinality four. Then, the following cases will occur.

Case 1: $\mathfrak{R} = \left\{x_0^a, x_{i_2}^a, x_{i_3}^a, x_{i_4}^a\right\}$.
Let us assume $i_r \in \left[\!\left[i_{r-1}+1, \min\left\{n-(4-r), i_{r-1}+\left\lfloor \frac{n}{2} \right\rfloor\right\}\right]\!\right]$, where $r \in \{2,3,4\}$. Then, by Table 1, for any $x_{i_2}^a, x_{i_3}^a, x_{i_4}^a \in V(CB_{4,n})$, there exists at least two vertices that are equidistant from all vertices in the set $\mathfrak{R}$.

Table 1: Equidistant vertices from $\mathfrak{R} = \left\{x_0^a, x_{i_2}^a, x_{i_3}^a, x_{i_4}^a\right\}$ with $i_2 \equiv a_2 \pmod 3$, $i_3 \equiv a_3 \pmod 3$ and $i_4 \equiv a_4 \pmod 3$.

$\mathbf{a_2}$	$\mathbf{a_3}$	$\mathbf{a_4}$	**Vertices having same metric representation**
$0,1$	$0,1$	$0,1$	$\{x_1^b, x_2^b\}$
$0,2$	$0,2$	$0,2$	$\{x_0^b, x_1^b\}$
$1,2$	$1,2$	$1,2$	$\{x_2^b, x_3^b\}$
$0,1$	2	$1,2$	$\{x_{i_3}^b, x_{i_3+1}^b\}$
0	1	2	$\{x_{i_3+1}^b, x_{i_3+2}^b\}$
2	0	1	
1	0	2	$\{x_{i_4}^b, x_{i_4+1}^b\}$
1	2	0	$\left\{x_{\lceil \frac{n}{2} \rceil - 1}^a, x_{\lceil \frac{n}{2} \rceil}^a\right\}$ or $\left\{x_{i_3+\lceil \frac{n}{2} \rceil - 1}^a, x_{i_3+\lceil \frac{n}{2} \rceil}^a\right\}$ when $n \equiv 1 \pmod 6$ $\left\{x_{i_2+\lceil \frac{n}{2} \rceil + 1}^b, x_{i_2+\lceil \frac{n}{2} \rceil + 2}^b\right\}$ when $n \equiv 4 \pmod 6$

Case 2: $\mathfrak{R} = \left\{x_0^a, x_{i_2}^a, x_{i_3}^a, x_{i_4}^b\right\}$.

Let us assume $i_r \in \left[\!\left[i_{r-1}+1, \min\left\{n-(3-r), i_{r-1}+\left\lfloor \frac{n}{2} \right\rfloor\right\}\right]\!\right]$, where $r \in \{2,3\}$ and $i_4 \in [\![0, n]\!]$. Then, by Tables 2 and 3, for any $x_{i_2}^a, x_{i_3}^a, x_{i_4}^b \in V(CB_{4,n})$, there exists at least two vertices that are equidistant from all vertices in the set $\mathfrak{R}$.

Table 2: Equidistant vertices from $\mathfrak{R} = \left\{x_0^a, x_{i_2}^a, x_{i_3}^a, x_{i_4}^b\right\}$ when $n \equiv 4 \pmod 6$, where $i_2 \equiv a_2 \pmod 3$, $i_3 \equiv a_3 \pmod 3$ and $i_4 \equiv a_4 \pmod 3$.

$\mathbf{a_2}$	$\mathbf{a_3}$	$\mathbf{a_4}$	**Vertices having same metric representation**
0	0	0	$\left\{x^b_{\frac{n}{2}}, x^b_{\frac{n}{2}-1}\right\}$
0	0	1	$\left\{x^b_{i_4-2}, x^b_{i_4-3}\right\}$
0	0	2	$\left\{x^b_{i_4+2}, x^b_{i_4+3}\right\}$
0	1	0	$\left\{x^b_{i_3}, x^b_{i_3+1}\right\}$
0	1	1	$\left\{x^b_{i_3+1}, x^b_{i_3+2}\right\}$
0	1	2	$\left\{x^a_{i_4+\frac{n}{2}-1}, x^a_{i_4+\frac{n}{2}-2}\right\}$ or $\left\{x^b_{i_3+1}, x^b_{i_3+2}\right\}$
0	2	0, 2	$\left\{x^b_{i_3+2}, x^b_{i_3+3}\right\}$ or $\left\{x^b_{i_4+3}, x^b_{i_4+3}\right\}$ or $\left\{x^b_{i_2+1}, x^b_{i_2+2}\right\}$
0	2	1	$\left\{x^b_{i_3}, x^b_{i_3+1}\right\}$
1	0	0	$\left\{x^b_{\max\{i_3,i_4\}+2}, x^b_{\max\{i_3,i_4\}+3}\right\}$
1	0	1	$\left\{x^a_{i_3+2}, x^a_{i_3+3}\right\}$ or $\left\{x^b_{\frac{n}{2}}, x^b_{\frac{n}{2}-1}\right\}$
1	0	2	$\left\{x^b_{i_2+\frac{n}{2}+1}, x^b_{i_2+\frac{n}{2}+2}\right\}$ or $\left\{x^a_{i_4-2}, x_{i_4-3}\right\}$ or $\left\{x^b_{i_2+1}, x^b_{i_2+2}\right\}$
1	1	0	$\left\{x^b_{i_4+2}, x^b_{i_4+3}\right\}$
1	1	1	$\left\{x^a_{i_4+1}, x^a_{i_4+2}\right\}$
1	1	2	$\left\{x^a_{i_4-2}, x^a_{i_4-3}\right\}$ or $\left\{x^a_{i_4+\frac{n}{2}-2}, x^a_{i_4+\frac{n}{2}-1}\right\}$
1	2	0	$\left\{x^b_{\frac{n}{2}+1}, x^b_{\frac{n}{2}+2}\right\}$ or $\left\{x^b_{i_3+\frac{n}{2}+1}, x^b_{i_3+\frac{n}{2}+2}\right\}$
1	2	1	$\left\{x^b_{i_3}, x^b_{i_3+1}\right\}$
1	2	2	$\left\{x^b_{i_2+\frac{n}{2}+1}, x^b_{i_2+\frac{n}{2}+2}\right\}$ or $\left\{x^b_{i_3}, x^b_{i_3+1}\right\}$
2	0	0	$\left\{x^b_{i_2+2}, x^b_{i_2+3}\right\}$
2	0	1	$\left\{x^b_{i_2+\frac{n}{2}+1}, x^b_{i_2+\frac{n}{2}+2}\right\}$ or $\left\{x^b_{i_3+\frac{n}{2}+1}, x^b_{i_3+\frac{n}{2}+2}\right\}$
2	0	2	$\left\{x^b_{\frac{n}{2}+2}, x^b_{\frac{n}{2}+3}\right\}$ or $\left\{x^b_{i_4+2}, x^b_{i_4+3}\right\}$

$\mathbf{a_2}$	$\mathbf{a_3}$	$\mathbf{a_4}$	**Vertices having same metric representation**
2	1	0, 1, 2	$\left\{x^b_{i_2}, x^b_{i_2+1}\right\}$ or $\left\{x^b_{i_3}, x^b_{i_3+1}\right\}$
2	2	0, 2	$\left\{x^b_{i_3+2}, x^b_{i_3+3}\right\}$ or $\left\{x^b_{i_4+2}, x^b_{i_4+3}\right\}$
2	2	1	$\left\{x^b_{\frac{n}{2}+1}, x^b_{\frac{n}{2}+2}\right\}$ or $\left\{x^b_{i_2}, x^b_{i_2+1}\right\}$

Table 3: Equidistant vertices from $\mathfrak{R} = \left\{x^a_0, x^a_{i_2}, x^a_{i_3}, x^b_{i_4}\right\}$ when $n \equiv 1 \pmod 6$, where $i_2 \equiv a_2 \pmod 3$, $i_3 \equiv a_3 \pmod 3$ and $i_4 \equiv a_4 \pmod 3$.

$\mathbf{a_2}$	$\mathbf{a_3}$	$\mathbf{a_4}$	**Vertices having same metric representation**
0	0	0	$\left\{x^a_{\frac{n}{2}}, x^a_{\frac{n}{2}+1}\right\}$
0	0	1	$\left\{x^b_{i_4-2}, x^b_{i_4-3}\right\}$ or $\left\{x^b_{i_2+2}, x^b_{i_2+1}\right\}$
0	0	2	$\left\{x^b_{i_4+2}, x^b_{i_4+3}\right\}$
0	1	0	$\left\{x^b_{i_3}, x^b_{i_3+1}\right\}$
0	1	1	$\left\{x^b_{i_3+1}, x^b_{i_3+2}\right\}$
0	1	2	$\left\{x^b_{i_4+\frac{n}{2}-1}, x^b_{i_4+\frac{n}{2}}\right\}$ or $\left\{x^b_{i_4-2}, x^b_{i_4-3}\right\}$
0	2	0	$\left\{x^a_{i_3+2}, x^a_{i_3+3}\right\}$ or $\left\{x^a_{i_2-1}, x^a_{i_2-2}\right\}$
0	2	1	$\left\{x^b_{i_4+\lceil\frac{n}{2}\rceil-1}, x^b_{i_4+\lceil\frac{n}{2}\rceil}\right\}$ or $\left\{x^b_{i_2+2}, x^b_{i_2+3}\right\}$
0	2	2	$\left\{x^b_0, x^b_1\right\}$
1	0	0	$\left\{x^b_{\max(i_3,i_4)+2}, x^b_{\max(i_3,i_4)+3}\right\}$
1	0	1	$\left\{x^b_{i_3+2}, x^b_{i_3+3}\right\}$
1	0, 1	2	$\left\{x^b_{i_4+\lceil\frac{n}{2}\rceil-1}, x^b_{i_4+\lceil\frac{n}{2}\rceil}\right\}$ or $\left\{x^b_{i_4-2}, x^b_{i_4-3}\right\}$ or $\left\{x^b_{i_4+2}, x^b_{i_4+3}\right\}$
1	1	0	$\left\{x^b_1, x^b_2\right\}$
1	1	1	$\left\{x^a_{i_4+1}, x^a_{i_4+2}\right\}$
1	1	2	$\left\{x^b_{i_4+\frac{n}{2}}, x^b_{i_4+\frac{n}{2}-1}\right\}$ or $\left\{x^a_{i_4+\frac{n}{2}-2}, x^a_{i_4+\frac{n}{2}-3}\right\}$
1	2	0, 1, 2	$\left\{x^a_{i_2+\lceil\frac{n}{2}\rceil-1}, x^a_{i_2+\lceil\frac{n}{2}\rceil}\right\}$ or $\left\{x^b_2, x^b_3\right\}$
2	0	1	$\left\{x^a_{i_2+\lceil\frac{n}{2}\rceil}, x^a_{i_2+\lceil\frac{n}{2}\rceil-1}\right\}$

$\mathbf{a_2}$	$\mathbf{a_3}$	$\mathbf{a_4}$	**Vertices having same metric representation**
2	0	0, 2	$\{x^b_{i_3+1}, x^b_{i_3+2}\}$ or $\{x^a_{i_4-1}, x^a_{i_4}\}$
2	1	0, 1, 2	$\{x^b_{i_2}, x^b_{i_2+1}\}$ or $\{x^b_{i_3}, x^b_{i_3+1}\}$
2	2	0, 2	$\{x^b_0, x^b_1\}$ or $\{x^b_{i_4+2}, x^b_{i_4+3}\}$
2	2	1	$\{x^b_{i_2}, x^b_{i_2+1}\}$ or $\{x^b_{i_4+2}, x^b_{i_4+3}\}$

Case 3: $\mathfrak{R} = \{x^a_0, x^a_{i_2}, x^b_{i_3}, x^b_{i_4}\}$.

Let us assume $i_2 \in \left[\!\left[1, \left\lfloor \frac{n}{2} \right\rfloor\right]\!\right]$, $i_3 \in [\![0, n-2]\!]$ and $i_4 \in \left[\!\left[i_3+1, \min\left\{n-1, i_3 + \left\lceil \frac{n}{2} \right\rceil\right\}\right]\!\right]$. Then, by Tables 4 and 5, for any $x^a_{i_2}, x^b_{i_3}, x^b_{i_4} \in V(CB_{4,n})$, there exists at least two vertices that are equidistant from all vertices in the set $\mathfrak{R}$.

Table 4: Equidistant vertices from $\mathfrak{R} = \{x^a_0, x^a_{i_2}, x^b_{i_3}, x^b_{i_4}\}$ when $n \equiv 1 \pmod 6$, where $i_2 \equiv a_2 \pmod 3$, $i_3 \equiv a_3 \pmod 3$ and $i_4 \equiv a_4 \pmod 3$.

$\mathbf{a_2}$	$\mathbf{a_3}$	$\mathbf{a_4}$	**Vertices having same metric representation**
0	0	0, 1	$\left\{x^a_{\lceil \frac{n}{2} \rceil}, x^a_{\lceil \frac{n}{2} \rceil+1}\right\}$ or $\left\{x^a_{\lceil \frac{n}{2} \rceil+1}, x^a_{\lceil \frac{n}{2} \rceil+2}\right\}$
0	0	2	$\left\{x^a_{\lceil \frac{n}{2} \rceil}, x^a_{\lceil \frac{n}{2} \rceil+1}\right\}$ or $\{x^b_{i_4+2}, x^b_{i_4+3}\}$
0	1	0	$\left\{x^b_{i_3+\lceil \frac{n}{2} \rceil-1}, x^b_{i_3+\lceil \frac{n}{2} \rceil}\right\}$ or $\{x^b_{i_4+2}, x^b_{i_4+3}\}$
0	1	1	$\left\{x^b_{i_3+\lceil \frac{n}{2} \rceil}, x^b_{i_3+\lceil \frac{n}{2} \rceil+1}\right\}$ or $\{x^b_{i_4+2}, x^b_{i_4+3}\}$
0	1	2	$\left\{x^b_{i_3+\lceil \frac{n}{2} \rceil-1}, x^b_{i_3+\lceil \frac{n}{2} \rceil}\right\}$ or $\{x^a_{i_2+2}, x^a_{i_2+3}\}$ or $\{x^a_{i_2-1}, x^a_{i_2+1}\}$
0	2	0	$\left\{x^b_{i_3+\lceil \frac{n}{2} \rceil-1}, x^b_{i_3+\lceil \frac{n}{2} \rceil}\right\}$ or $\{x^b_{i_4-1}, x^b_{i_4-2}\}$
0	2	1	$\left\{x^b_{i_3+\lceil \frac{n}{2} \rceil-1}, x^b_{i_3+\lceil \frac{n}{2} \rceil}\right\}$
0	2	2	$\left\{x^a_{\lceil \frac{n}{2} \rceil}, x^a_{\lceil \frac{n}{2} \rceil+1}\right\}$ or $\{x^b_{i_4+2}, x^b_{i_4+3}\}$
1	0	0	$\{x^b_{i_4+2}, x^b_{i_4+3}\}$
1	0, 1	1	$\{x^b_{i_4+1}, x^b_{i_4+2}\}$
1	0	2	$\left\{x^b_{i_4+\lceil \frac{n}{2} \rceil-1}, x^b_{i_4+\lceil \frac{n}{2} \rceil}\right\}$ or $\{x^a_{i_3}, x^a_{i_3-1}\}$
1	1	0	$\{x^a_{i_4}, x^a_{i_4-1}\}$
1	1	2	$\left\{x^b_{i_4+\lceil \frac{n}{2} \rceil-1}, x^b_{i_4+\lceil \frac{n}{2} \rceil}\right\}$ or $\{x^a_{i_4-2}, x^a_{i_4-3}\}$

a_2	a_3	a_4	**Vertices having same metric representation**
1	2	0	$\left\{x^b_{i_3+\lceil\frac{n}{2}\rceil-1}, x^b_{i_3+\lceil\frac{n}{2}\rceil}\right\}$ or $\left\{x^a_{\lceil\frac{n}{2}\rceil-1}, x^a_{\lceil\frac{n}{2}\rceil}\right\}$ or $\left\{x^b_{i_3+2}, x^b_{i_3+3}\right\}$
1	2	1	$\left\{x^b_{i_3+\lceil\frac{n}{2}\rceil-1}, x^b_{i_3\lceil\frac{n}{2}\rceil}\right\}$ or $\left\{x^b_{i_3+\lceil\frac{n}{2}\rceil}, x^b_{i_3\lceil\frac{n}{2}\rceil+1}\right\}$
1	2	2	$\left\{x^a_{\lceil\frac{n}{2}\rceil}, x^a_{\lceil\frac{n}{2}\rceil-1}\right\}$ or $\left\{x^a_1, x^a_{n-1}\right\}$
2	0	0	$\left\{x^a_{i_2+2}, x^a_{i_2+3}\right\}$
2	0	1	$\left\{x^a_{\lceil\frac{n}{2}\rceil-2}, x^a_{\lceil\frac{n}{2}\rceil-3}\right\}$ or $\left\{x^a_{i_2+\lceil\frac{n}{2}\rceil-1}, x^a_{i_2+\lceil\frac{n}{2}\rceil}\right\}$ or $\left\{x^a_{i_3-1}, x^a_{i_3-2}\right\}$
2	0	2	$\left\{x^a_{i_2+\lceil\frac{n}{2}\rceil-1}, x^a_{i_2+\lceil\frac{n}{2}\rceil-2}\right\}$ or $\left\{x^b_{i_4+\lceil\frac{n}{2}\rceil-1}, x^b_{i_4+\lceil\frac{n}{2}\rceil}\right\}$
2	1	$0,1$	$\left\{x^a_{i_2+\lceil\frac{n}{2}\rceil-1}, x^a_{i_2+\lceil\frac{n}{2}\rceil}\right\}$ or $\left\{x^b_{i_4+\lceil\frac{n}{2}\rceil-1}, x^b_{i_4+\lceil\frac{n}{2}\rceil-2}\right\}$
2	$1,2$	$1,2$	$\left\{x^a_{\lceil\frac{n}{2}\rceil-1}, x^a_{\lceil\frac{n}{2}\rceil}\right\}$ or $\left\{x^b_2, x^b_3\right\}$
2	2	0	$\left\{x^a_{\lceil\frac{n}{2}\rceil+1}, x^a_{\lceil\frac{n}{2}\rceil}\right\}$ or $\left\{x^a_{\lceil\frac{n}{2}\rceil-1}, x^a_{\lceil\frac{n}{2}\rceil}\right\}$

Table 5: Equidistant vertices from $\mathfrak{R} = \left\{x^a_0, x^a_{i_2}, x^b_{i_3}, x^b_{i_4}\right\}$ when $n \equiv 4 \pmod 6$, where $i_2 \equiv a_2 \pmod 3$, $i_3 \equiv a_3 \pmod 3$ and $i_4 \equiv a_4 \pmod 3$.

a_2	a_3	a_4	**Vertices having same metric representation**
0	0	0	$\left\{x^a_{i_2+1}, x^a_{i_2+2}\right\}$
0	0	1	$\left\{x^b_{i_2+\frac{n}{2}}, x^b_{i_2+\frac{n}{2}+1}\right\}$ or $\left\{x^b_{i_2+\frac{n}{2}}, x^b_{i_2+\frac{n}{2}-1}\right\}$
0	0	2	$\left\{x^a_{i_4}, x^a_{i_4-1}\right\}$
0	1	0	$\left\{x^a_{i_3-\frac{n}{2}-1}, x^a_{i_3-\frac{n}{2}-2}\right\}$ or $\left\{x^a_{i_3-\frac{n}{2}}, x^a_{i_3-\frac{n}{2}-1}\right\}$
0	1	1	$\left\{x^a_{i_3-\frac{n}{2}-1}, x^a_{i_3-\frac{n}{2}-2}\right\}$ or $\left\{x^a_{i_3-\frac{n}{2}-1}, x^a_{i_3-\frac{n}{2}}\right\}$
0	1	2	$\left\{x^a_{i_4-\frac{n}{2}-1}, x^a_{i_4-\frac{n}{2}-2}\right\}$ or $\left\{x^a_{i_4-\frac{n}{2}-1}, x^a_{i_4-\frac{n}{2}}\right\}$
0	2	$0,1$	$\left\{x^a_{i_3-\frac{n}{2}+1}, x^a_{i_3-\frac{n}{2}}\right\}$ or $\left\{x^a_{i_3-\frac{n}{2}-1}, x^a_{i_3-\frac{n}{2}-2}\right\}$
0	2	2	$\left\{x^a_{i_3-\frac{n}{2}-2}, x^a_{i_3-\frac{n}{2}-3}\right\}$ or $\left\{x^a_{i_2+2}, x^a_{i_2+3}\right\}$
1	0	0	$\left\{x^a_{i_4}, x^a_{i_4-1}\right\}$
1	0	1	$\left\{x^b_{i_4+1}, x^a_{i_4+2}\right\}$
1	0	2	$\left\{x^a_{i_4-\frac{n}{2}-1}, x^a_{i_4-\frac{n}{2}-2}\right\}$ or $\left\{x^a_{i_3}, x^a_{i_3-1}\right\}$

$\mathbf{a_2}$	$\mathbf{a_3}$	$\mathbf{a_4}$	**Vertices having same metric representation**
1	1	0	$\left\{x^a_{i_4-\frac{n}{2}-3}, x^a_{i_4-\frac{n}{2}-2}\right\}$ or $\left\{x^a_{i_4-\frac{n}{2}-1}, x^a_{i_4-\frac{n}{2}-2}\right\}$
1	1	1	$\left\{x^a_{i_4-\frac{n}{2}-1}, x^a_{i_4-\frac{n}{2}}\right\}$ or $\left\{x^a_{i_4-\frac{n}{2}}, x^a_{i_4-\frac{n}{2}+1}\right\}$
1	1	2	$\left\{x^a_{i_4-\frac{n}{2}-1}, x^a_{i_4-\frac{n}{2}-2}\right\}$ or $\left\{x^a_{i_4-\frac{n}{2}-1}, x^a_{i_4-\frac{n}{2}}\right\}$
1	2	0	$\left\{x^a_{i_3-\frac{n}{2}-1}, x^a_{i_3-\frac{n}{2}-2}\right\}$ or $\left\{x^a_{i_4-\frac{n}{2}-2}, x^a_{i_4-\frac{n}{2}-1}\right\}$
1	2	1	$\left\{x^a_{i_3-\frac{n}{2}-1}, x^a_{i_3-\frac{n}{2}-2}\right\}$ or $\left\{x^a_{i_4-\frac{n}{2}-2}, x^a_{i_4-\frac{n}{2}-3}\right\}$
1	2	2	$\left\{x^b_{i_2+\frac{n}{2}}, x^b_{i_2+\frac{n}{2}+1}\right\}$ or $\left\{x^b_{\frac{n}{2}+1}, x^b_{\frac{n}{2}+2}\right\}$
2	0	0	$\left\{x^a_{i_2+2}, x^a_{i_2+3}\right\}$
2	0	1	$\left\{x^b_{i_2+\frac{n}{2}+2}, x^b_{i_2+\frac{n}{2}+1}\right\}$ or $\left\{x^b_2, x^b_3\right\}$ or $\left\{x^a_{i_3-1}, x^a_{i_3-2}\right\}$
2	0	2	$\left\{x^b_{\frac{n}{2}}, x^b_{\frac{n}{2}-1}\right\}$ or $\left\{x^a_{n-2}, x^a_{n-3}\right\}$
2	1	0	$\left\{x^a_{i_3+\frac{n}{2}-1}, x^a_{i_2+\frac{n}{2}-2}\right\}$ or $\left\{x^a_{i_2+2}, x^a_{i_2+3}\right\}$ or $\left\{x^b_{i_3-1}, x^b_{i_3+1}\right\}$
2	1	1	$\left\{x^b_{i_2+\frac{n}{2}+1}, x^b_{i_2+\frac{n}{2}+2}\right\}$ or $\left\{x^b_2, x^b_3\right\}$
2	1	2	$\left\{x^a_{i_3+\frac{n}{2}+-1}, x^a_{i_3+\frac{n}{2}-2}\right\}$ or $\left\{x^b_{\frac{n}{2}+1}, x^b_{\frac{n}{2}+2}\right\}$ or $\left\{x^b_0, x^b_1\right\}$
2	2	0	$\left\{x^b_{\frac{n}{2}}, x^b_{\frac{n}{2}-1}\right\}$ or $\left\{x^b_{i_4+1}, x^b_{i_4+2}\right\}$ or $\left\{x^b_0, x^b_1\right\}$
2	2	1, 2	$\left\{x^a_{i_4+\frac{n}{2}-1}, x^a_{i_4+\frac{n}{2}-2}\right\}$ or $\left\{x^b_2, x^b_3\right\}$ or $\left\{x^a_{n-1}, x^a_1\right\}$

From the above cases, we can conclude that $\dim(CB_{4,n}) \geq 5$, for any $n \geq 7$ and $n \equiv 1 \pmod 3$. □

The following theorem determines the exact value of $\dim(CB(4,n))$ for $n \not\equiv 1 \pmod 3$ by using the lower bound established in the Lemma 3.1 and presenting a resolving set that attains this bound.

Theorem 3.1. *For any $n \geq 5$ and $n \not\equiv 1 \pmod 3$, $\dim(CB_{4,n}) = 4$.*

Proof. We can conclude that $\dim(CB_{4,n}) \geq 4$ by Lemma 3.1. To prove that $\dim(CB_{4,n}) \leq 4$, we can give a resolving set of cardinality equal to four for the graph $CB_{4,n}$. Consider $W = \left\{x^a_0, x^a_1, x^b_1, x^b_2\right\}$ to be the resolving set of $CB_{4,n}$, where $n \not\equiv 1 \pmod 3$. Obviously, every vertex in W gets resolved. So, we consider only the vertices that are not in W.

First, we can prove that every vertex of X^a not in W have a unique metric representation. For the vertices x_i^a, where $i \left(\text{mod} \left\lfloor \frac{n}{2} \right\rfloor\right) \equiv 1 \pmod 3$, we have

$$\begin{aligned} d(x_0^a, x_i^a) &= d(x_1^a, x_i^a) + 2, \text{ when } i \leq \left\lfloor \frac{n}{2} \right\rfloor, \\ d(x_0^a, x_i^a) &= d(x_1^a, x_i^a) - 2, \text{ when } i > \left\lfloor \frac{n}{2} \right\rfloor, \end{aligned} \tag{1}$$

implies that the vertices x_i^a, where $i \left(\text{mod} \left\lfloor \frac{n}{2} \right\rfloor\right) \equiv 1 \pmod 3$ have distinct metric representations.

For the vertices x_i^a, where $i \left(\text{mod} \left\lfloor \frac{n}{2} \right\rfloor\right) \not\equiv 1 \pmod 3$, we have

$$d(x_0^a, x_i^a) = d(x_1^a, x_i^a). \tag{2}$$

But, for the vertices x_i^a, where $i \left(\text{mod} \left\lfloor \frac{n}{2} \right\rfloor\right) \equiv 2 \pmod 3$, we have

$$\begin{aligned} d(x_1^b, x_i^a) &= d(x_2^b, x_i^a) + 2, \text{ when } i \leq \left\lfloor \frac{n}{2} \right\rfloor, \\ d(x_1^b, x_i^a) &= d(x_2^b, x_i^a) - 2, \text{ when } i > \left\lfloor \frac{n}{2} \right\rfloor, \end{aligned} \tag{3}$$

and for the vertices x_i^a, where $i \left(\text{mod} \left\lfloor \frac{n}{2} \right\rfloor\right) \equiv 0 \pmod 3$, we have

$$d(x_1^b, x_i^a) = d(x_2^b, x_i^a). \tag{4}$$

Equations (1), (2), (3) and (4) implies that every vertex $x_i^a \in X^a$, where $0 \leq i \leq n-1$, have distinct metric representations. Now we consider the vertices in the partition X^b.

For the vertices x_i^b, where $i - 1 \left(\text{mod} \left\lfloor \frac{n}{2} \right\rfloor\right) \equiv 1 \pmod 3$, we have

$$\begin{aligned} d(x_0^b, x_i^b) &= d(x_1^b, x_i^b) + 2, \text{ when } i - 1 (\text{mod } n) \leq \left\lfloor \frac{n}{2} \right\rfloor + 1, \\ d(x_0^b, x_i^b) &= d(x_1^b, x_i^b) - 2, \text{ when } i - 1 (\text{mod } n) > \left\lfloor \frac{n}{2} \right\rfloor + 1, \end{aligned} \tag{5}$$

implies that the vertices x_i^b, where $i - 1 \left(\text{mod} \left\lfloor \frac{n}{2} \right\rfloor\right) \equiv 1 \pmod 3$, have distinct metric representations.

For the vertices x_i^b where, $i - 1 \left(\text{mod} \left\lfloor \frac{n}{2} \right\rfloor\right) \not\equiv 1 \pmod 3$, we have

$$d(x_0^b, x_i^b) = d(x_1^b, x_i^b). \tag{6}$$

But, for the vertices x_i^b, where $i - 1 \left(\text{mod} \left\lfloor \frac{n}{2} \right\rfloor\right) \equiv 0 \pmod 3$, we have

$$\begin{aligned} d(x_0^a, x_i^b) &= d(x_1^a, x_i^b) + 2, \text{ when } i - 1 (\text{mod } n) \leq \left\lfloor \frac{n}{2} \right\rfloor + 1, \\ d(x_0^a, x_i^b) &= d(x_1^a, x_i^b) - 2, \text{ when } i - 1 (\text{mod } n) > \left\lfloor \frac{n}{2} \right\rfloor + 1, \end{aligned} \tag{7}$$

and for the vertices x_i^a, where $i - 1 \left(\text{mod} \left\lfloor \frac{n}{2} \right\rfloor\right) \equiv 2 \pmod 3$, we have

$$d\left(x_0^a, x_i^b\right) = d\left(x_1^a, x_i^b\right). \tag{8}$$

Equations (5), (6), (7) and (8) implies that every vertex $x_i^a \in X^b$, where $0 \leq i \leq n-1$, have distinct metric representations. Hence, W is the resolving set for $CB_{4,n}$, implies that, $\dim(CB_{4,n}) = 4$, when $n \not\equiv 1 \pmod 3$. □

Theorem 3.2. *For $n \geq 5$ and $n \equiv 1 \pmod 3$, $\dim(CB_{4,n}) = 5$.*

Proof. By Lemma 3.2, $\dim(CB_{4,n}) \geq 5$, when $n \equiv 1 \pmod 3$ can be concluded. It is sufficient to prove that $\dim(CB_{4,n}) \leq 5$, when $n \equiv 1 \pmod 3$. For that, it is enough to show the existence of a resolving set W, $|W| = 5$ exists. Now, let us assume that $\mathfrak{R} = \left\{x_0^a, x_2^a, x_1^b, x_3^b, x^b_{3\lfloor \frac{n-3}{6} \rfloor+5}\right\}$ is a resolving set of $CB_{4,n}$, where $n \equiv 1 \pmod 3$.

Case 1: $n \equiv 1 (\text{mod } 6)$

The metric representation of $x \in X^a \backslash \mathfrak{R}$, with respect to $\mathfrak{R} = \left\{x_0^a, x_2^a, x_1^b, x_3^b, x^b_{3\lfloor \frac{n-3}{6} \rfloor+5}\right\}$ is given by the following equations:

$$r(x_1^a|\mathfrak{R}) = \left(2, 2, 1, 1, \frac{n+2}{3}\right),$$

$$r(x_{3j}^a|\mathfrak{R}) = \left(2j, 2j, 2j+1, 2j-1, \left(\frac{n+2}{3}\right) - 2j\right),\ j \in \left[\!\left[1, \frac{n-1}{6}\right]\!\right],$$

$$r(x_{3j+1}^a|\mathfrak{R}) = \left(2j+2, 2j, 2j+1, 2j+1, \left(\frac{n+2}{3}\right) - 2j\right),\ j \in \left[\!\left[1, \frac{n-7}{6}\right]\!\right],$$

$$r(x_{3j+2}^a|\mathfrak{R}) = \left(2j+2, 2j, 2j+3, 2j+1, \left(\frac{n-4}{3}\right) - 2j\right),\ j \in \left[\!\left[1, \frac{n-7}{6}\right]\!\right],$$

$$r\left(x^a_{\frac{n-1}{2}+3j+1}|\mathfrak{R}\right) = \left(2\left(\frac{n-1}{6} - j\right), 2\left(\frac{n-1}{6} - j\right), 2\left(\frac{n-1}{6} - j\right) + 1, 2\left(\frac{n-1}{6} - j\right) + 1, 2j+1\right),\ j \in \left[\!\left[0, \frac{n-7}{6}\right]\!\right],$$

$$r\left(x^a_{\frac{n-1}{2}+3j+2}|\mathfrak{R}\right) = \left(2\left(\frac{n-1}{6} - j\right), 2\left(\frac{n-1}{6} - j\right), 2\left(\frac{n-1}{6} - j\right) - 1, 2\left(\frac{n-1}{6} - j\right) + 1, 2j+1\right),\ j \in \left[\!\left[0, \frac{n-7}{6}\right]\!\right],$$

$$r\left(x^a_{\frac{n-1}{2}+3j+3}|\mathfrak{R}\right) = \left(2\left(\frac{n-1}{6} - j\right), 2\left(\frac{n-1}{6} - j\right), 2\left(\frac{n-1}{6} - j\right) - 1, 2\left(\frac{n-1}{6} - j\right) + 1, 2j+3\right),\ j \in \left[\!\left[0, \frac{n-7}{6}\right]\!\right].$$

This confirms that no two vertices in set X^a receive identical metric representation with respect to $\mathfrak{R}$. Also, the following equations provide the metric representation of any $x \in X^b \backslash \mathfrak{R}$:

$$r\left(x_0^b|\mathfrak{R}\right) = \left(1, 3, 2, 2, \frac{n-1}{3}\right),$$

$$r\left(x_2^b|\mathfrak{R}\right) = \left(1, 1, 2, 2, \frac{n-1}{3}\right),$$

$$r\left(x_{3j+1}^b|\mathfrak{R}\right) = \left(2j+1, 2j-1, 2j, 2j, \left(\frac{n-1}{3}\right) - 2(j-1)\right),\ j \in \left[\!\left[1, \frac{n-1}{6}\right]\!\right],$$

$$r\left(x_{3j+2}^b|\mathfrak{R}\right) = \left(2j+1, 2j-1, 2j+2, 2j, \left(\frac{n-1}{3}\right) - 2j\right),\ j \in \left[\!\left[1, \frac{n-7}{6}\right]\!\right],$$

$$r\left(x^b_{3j+3}|\mathfrak{R}\right) = \left(2j+1, 2j+1, 2j+2, 2j, \left(\frac{n-1}{3}\right) - 2j\right),\ j \in \left[\!\left[1, \frac{n-7}{6}\right]\!\right],$$

$$r\left(x^b_{\frac{n+5}{2}}|\mathfrak{R}\right) = \left(\frac{n+2}{3}, \frac{n+2}{3}, \frac{n-1}{3}, \frac{n-1}{3}, 2\right),$$

$$r\left(x^a_{\frac{n-1}{2}+3j+1}|\mathfrak{R}\right) = \left(\frac{n+1}{3} - 2j, \frac{n+1}{3} - 2(j-1), \frac{n-1}{3} - 2(j-1), \frac{n-1}{3} - 2(j-1), 2j\right),\ j \in \left[\!\left[0, \frac{n-7}{6}\right]\!\right],$$

$$r\left(x^a_{\frac{n-1}{2}+3j+2}|\mathfrak{R}\right) = \left(\frac{n+1}{3} - 2j, \frac{n+1}{3} - 2(j-1), \frac{n-1}{3} - 2j, \frac{n-1}{3} - 2(j-1), 2j\right),\ j \in \left[\!\left[0, \frac{n-7}{6}\right]\!\right],$$

$$r\left(x^a_{\frac{n-1}{2}+3j+3}|\mathfrak{R}\right) = \left(\frac{n+1}{3} - 2j, \frac{n+1}{3} - 2j, \frac{n-1}{3} - 2j, \frac{n-1}{3} - 2(j-1), 2j+2\right),\ j \in \left[\!\left[0, \frac{n-7}{6}\right]\!\right].$$

This shows that no two vertices in set X^b have the same metric representation with respect to $\mathfrak{R}$. Hence, $\mathfrak{R}$ is a resolving set for $CB_{4,n}$, when $n \equiv 1 \pmod 6$.

Case 2: $n \equiv 4 \pmod 6$

The metric representation of any $x \in X^a \backslash \mathfrak{R}$, with respect to $\mathfrak{R} = \left\{x^a_0, x^a_2, x^b_1, x^b_3, x^b_{3\lfloor \frac{n-3}{6} \rfloor+5}\right\}$ is given by the following equations:

$$r(x^a_1|\mathfrak{R}) = \left(2, 2, 1, 1, \frac{n+2}{3}\right),$$

$$r(x^a_{3j}|\mathfrak{R}) = \left(2j, 2j, 2j+1, 2j-1, \left(\frac{n+5}{3}\right) - 2j\right),\ j \in \left[\!\left[1, \frac{n-4}{6}\right]\!\right],$$

$$r(x^a_{3j+1}|\mathfrak{R}) = \left(2j+2, 2j, 2j+1, 2j+1, \left(\frac{n+5}{3}\right) - 2j\right),\ j \in \left[\!\left[1, \frac{n-4}{6}\right]\!\right],$$

$$r(x^a_{3j+2}|\mathfrak{R}) = \left(2j+2, 2j, 2j+3, 2j+1, \left(\frac{n-1}{3}\right) - 2j\right),\ j \in \left[\!\left[1, \frac{n-10}{6}\right]\!\right],$$

$$r(x^a_{\frac{n+2}{2}}|\mathfrak{R}) = \left(\frac{n+2}{3}, \frac{n+2}{3}, \frac{n+2}{3} - 1, \frac{n+2}{3} - 1, 1\right),$$

$$r\left(x^a_{\frac{n-1}{2}+3j+1}|\mathfrak{R}\right) = \left(\frac{n-4}{3} - 2j, \frac{n+2}{3} - 2j, \frac{n+2}{3} - 2j + 1, \frac{n+2}{3} - 2j + 1, 2j+1\right),\ j \in \left[\!\left[0, \frac{n-10}{6}\right]\!\right],$$

$$r\left(x^a_{\frac{n-1}{2}+3j+2}|\mathfrak{R}\right) = \left(\frac{n-4}{3} - 2j, \frac{n+2}{3} - 2j, \frac{n+2}{3} - 2j - 1, \frac{n+2}{3} - 2j + 1, 2j+1\right),\ j \in \left[\!\left[0, \frac{n-10}{6}\right]\!\right],$$

$$r\left(x^a_{\frac{n-1}{2}+3j+2}|\mathfrak{R}\right) = \left(\frac{n-4}{3} - 2j, \frac{n+2}{3} - 2(j+1), \frac{n+2}{3} - 2j - 1, \frac{n+2}{3} - 2j + 1, 2j+3\right),$$
$$j \in \left[\!\left[0, \frac{n-10}{6}\right]\!\right].$$

This confirms that no two vertices in partition X^a have same metric representation with respect to $\mathfrak{R}$. Also, the following equations provide the metric representation of any $x \in X^b$:

$$r\left(x^b_0|\mathfrak{R}\right) = \left(1, 3, 2, 2, \frac{n-4}{3}\right),$$
$$r\left(x^b_2|\mathfrak{R}\right) = \left(1, 1, 2, 2, \frac{n+2}{3}\right),$$
$$r\left(x^b_{3j+1}|\mathfrak{R}\right) = \left(2j+1, 2j-1, 2j, 2j, \left(\frac{n-2}{2}\right) - 2j\right), \ j \in \left[\!\left[1, \frac{v-4}{6}\right]\!\right],$$
$$r\left(x^b_{3j+2}|\mathfrak{R}\right) = \left(2j+1, 2j-1, 2j+2, 2j, \left(\frac{n+2}{2}\right) - 2j\right), \ j \in \left[\!\left[1, \frac{n-4}{6}\right]\!\right],$$
$$r\left(x^b_{3j+3}|\mathfrak{R}\right) = \left(2j+1, 2j+1, 2j+2, 2j, \left(\frac{n+2}{2}\right) - 2j\right), \ j \in \left[\!\left[1, \frac{n-4}{6}\right]\!\right],$$
$$r\left(x^b_{\frac{n+4}{2}}|\mathfrak{R}\right) = \left(\frac{n-1}{3}, \frac{n-1}{3}, \frac{n+2}{3}, \frac{n+2}{3}, 2\right),$$
$$r\left(x^b_{\frac{n}{2}+3j}|\mathfrak{R}\right) = \left(\frac{n-1}{3} - 2j, \frac{n-1}{3} - 2j, \frac{n-4}{3} - 2j, \frac{n+2}{3} - 2j, 2j\right),$$
$$j \in \left[\!\left[0, \frac{n-4}{6}\right]\!\right],$$
$$r\left(x^b_{\frac{n}{2}+3j+1}|\mathfrak{R}\right) = \left(\frac{n-1}{3} - 2j, \frac{n-1}{3} - 2j, \frac{n-4}{3} - 2j, \frac{n+2}{3} - 2j, 2j+2\right),$$
$$j \in \left[\!\left[0, \frac{n-4}{6}\right]\!\right],$$
$$r\left(x^b_{\frac{n}{2}+3j+2}|\mathfrak{R}\right) = \left(\frac{n-4}{3} - 2j, \frac{n-1}{3} - 2j, \frac{n-4}{3} - 2j, \frac{n-4}{3} - 2j, 2j+2\right),$$
$$j \in \left[\!\left[0, \frac{n-4}{6}\right]\!\right].$$

This shows that no two vertices in the partition X^b have the same metric representation with respect to $\mathfrak{R}$. Hence, $\mathfrak{R}$ is a resolving set for $CB_{4,n}$, when $n \equiv 4 \pmod 6$.

From the above cases, it is concluded that $\dim(CB_{4,n}) = 5$, when $n \equiv 1 \pmod 3$. □

4 Conclusion

In this paper, we have determined the exact metric dimension of 4-regular cyclic bipartite graphs $CB_{4,n}$, that is $\dim(CB_{4,n}) = 4$ when $n \not\equiv 1 \pmod 3$ and $\dim(CB_{4,n}) = 5$ when $n \equiv 1 \pmod 3$ and $n \geq 5$. These results highlight a key bounded behavior of the metric dimension even as the graph order increases, which has practical implications for applications in network topology, sensor deployment, and fault detection in cyclic structures.

Our findings contribute both to the theoretical framework of resolving sets in symmetric graphs and to practical problems where efficient vertex identification is crucial. A promising direction for future research is to generalize this study to k-regular cyclic bipartite graphs for $k > 4$. Investigating how the metric dimension varies with increasing degree and identifying structural thresholds can further deepen the understanding of resolving sets in regular and symmetric graph families.

Acknowledgement All data generated or analysed during this study are included in this published article. The authors declare that there is no funding agency that supports this research.

Conflicts of Interest The authors declare no conflict of interest.

References

[1] S. Akhter & R. Farooq (2019). Metric dimension of fullerene graphs. *Electronic Journal of Graph Theory and Applications*, *7*(1), 91–103. http://dx.doi.org/10.5614/ejgta.2019.7.1.7.

[2] M. Arulperumjothi, S. Klavžar & S. Prabhu (2023). Redefining fractal cubic networks and determining their metric dimension and fault-tolerant metric dimension. *Applied Mathematics and Computation*, *452*, Article ID: 128037. https://doi.org/10.1016/j.amc.2023.128037.

[3] M. Azeem & M. K. Jamil (2024). Constant partition dimension of different anticancer drug structures. *Spectrum of Decision Making and Applications*, *1*(1), 64–83. https://doi.org/10.31181/sdmap1120245.

[4] M. Azeem & M. K. Jamil (2024). On the anticancer drug structures and their locating numbers. *Spectrum of Operational Research*, *1*(1), 44–63. https://doi.org/10.31181/sor1120245.

[5] M. Azeem & M. F. Nadeem (2021). Metric-based resolvability of polycyclic aromatic hydrocarbons. *The European Physical Journal Plus*, *136*(4), 1–14. https://doi.org/10.1140/epjp/s13360-021-01399-8.

[6] M. Bača, E. T. Baskoro, A. Salman, S. W. Saputro & D. Suprijanto (2011). The metric dimension of regular bipartite graphs. *Bulletin Mathématique de la Société des Sciences Mathématiques de Roumanie*, *54*(*102*)(1), 15–28.

[7] R. F. Bailey (2018). On the metric dimension of incidence graphs. *Discrete Mathematics*, *341*(6), 1613–1619. https://doi.org/10.1016/j.disc.2018.03.001.

[8] R. F. Bailey, J. Cáceres, D. Garijo, A. González, A. Márquez, K. Meagher & M. L. Puertas (2013). Resolving sets for Johnson and Kneser graphs. *European Journal of Combinatorics*, *34*(4), 736–751. https://doi.org/10.1016/j.ejc.2012.10.008.

[9] R. F. Bailey & K. Meagher (2012). On the metric dimension of grassmann graphs. *Discrete Mathematics & Theoretical Computer Science*, *13*(4), 97–104. https://doi.org/10.46298/dmtcs.532.

[10] A. F. Beardon (2013). Resolving the hypercube. *Discrete Applied Mathematics*, *161*(13-14), 1882–1887. https://doi.org/10.1016/j.dam.2013.02.012.

[11] Z. Beerliova, F. Eberhard, T. Erlebach, A. Hall, M. Hoffmann, M. Mihalák & L. S. Ram (2006). Network discovery and verification. *IEEE Journal on Selected Areas In Communications*, *24*(12), 2168–2181. https://doi.org/10.1109/JSAC.2006.884015.

[12] A. Behtoei, A. Davoodi, M. Jannesari & B. Omoomi (2017). A characterization of some graphs with metric dimension two. *Discrete Mathematics, Algorithms and Applications*, *9*(02), Article ID: 1750027. https://doi.org/10.1142/S1793830917500276.

[13] G. Brito, I. Dumitriu & K. D. Harris (2022). Spectral gap in random bipartite biregular graphs and applications. *Combinatorics, Probability and Computing*, *31*(2), 229–267. https://doi.org/10.1017/S0963548321000249.

[14] S. Bukhari, M. K. Jamil, M. Azeem & S. Swaray (2024). Honeycomb rhombic torus vertex-edge based resolvability parameters and its application in robot navigation. *IEEE Access*, *12*, 23751–23757. https://doi.org/10.1109/ACCESS.2024.3359916.

[15] J. Cáceres, C. Hernando, M. Mora, I. M. Pelayo & M. L. Puertas (2009). On the metric dimension of infinite graphs. *Electronic Notes in Discrete Mathematics*, *35*, 15–20. https://doi.org/10.1016/j.endm.2009.11.004.

[16] J. Cáceres, C. Hernando, M. Mora, I. M. Pelayo, M. L. Puertas, C. Seara & D. R. Wood (2007). On the metric dimension of cartesian products of graphs. *SIAM Journal on Discrete Mathematics*, *21*(2), 423–441. https://doi.org/10.1137/050641867.

[17] G. Chartrand, L. Eroh, M. A. Johnson & O. R. Oellermann (2000). Resolvability in graphs and the metric dimension of a graph. *Discrete Applied Mathematics*, *105*(1-3), 99–113. https://doi.org/10.1016/S0166-218X(00)00198-0.

[18] M. Fehr, S. Gosselin & O. R. Oellermann (2006). The metric dimension of Cayley digraphs. *Discrete Mathematics*, *306*(1), 31–41. https://doi.org/10.1016/j.disc.2005.09.015.

[19] C. Grigorious, P. Manuel, M. Miller, B. Rajan & S. Stephen (2014). On the metric dimension of circulant and Harary graphs. *Applied Mathematics and Computation*, *248*, 47–54. https://doi.org/10.1016/j.amc.2014.09.045.

[20] J. L. Guillaume & M. Latapy (2004). Bipartite structure of all complex networks. *Information Processing Letters*, *90*(5), 215–221. https://doi.org/10.1016/j.ipl.2004.03.007.

[21] M. Hallaway, C. X. Kang & E. Yi (2014). On metric dimension of permutation graphs. *Journal of Combinatorial Optimization*, *28*(4), 814–826. https://doi.org/10.1007/s10878-012-9587-3.

[22] F. Harary & R. A. Melter (1976). On the metric dimension of a graph. *Ars Combinatoria*, *2*, 191–195.

[23] R. Hasni & Y. H. Peng (2007). Chromatically unique bipartite graphs with certain 3-independent partition numbers III. *Malaysian Journal of Mathematical Sciences*, *1*(1), 139–162.

[24] C. Hernando, M. Mora, I. M. Pelayo, C. Seara & D. R. Wood (2007). Extremal graph theory for metric dimension and diameter. *Electronic Notes in Discrete Mathematics*, *29*, 339–343. https://doi.org/10.1016/j.endm.2007.07.058.

[25] M. Imran, A. Q. Baig, S. A. U. H. Bokhary & I. Javaid (2012). On the metric dimension of circulant graphs. *Applied Mathematics Letters*, *25*(3), 320–325. https://doi.org/10.1016/j.aml.2011.09.008.

[26] R. Ismail, S. Ali, M. Azeem & M. A. Zahid (2024). Double resolvability parameters of fosmidomycin anti-malaria drug and exchange property. *Heliyon*, *10*(13), Article ID: e33211. https://doi.org/10.1016/j.heliyon.2024.e33211.

[27] Y. Jiang & M. Lu (2019). Game total domination for cyclic bipartite graphs. *Discrete Applied Mathematics*, *265*, 120–127. https://doi.org/10.1016/j.dam.2019.03.009.

[28] M. Johnson (1993). Structure-activity maps for visualizing the graph variables arising in drug design. *Journal of Biopharmaceutical Statistics*, *3*(2), 203–236. https://doi.org/10.1080/10543409308835060.

[29] P. Karthikeyan, K. Kalaiselvi, M. Pachamuthu, K. Karthik & A. J. Santhosh (2023). An optimisation method of construction for warping copper plates and engines using complete block designs with some special types of graphs. *Advances in Materials Science and Engineering*, *2023*(1), Article ID: 9002743. https://doi.org/10.1155/2023/9002743.

[30] S. Khuller, B. Raghavachari & A. Rosenfeld (1996). Landmarks in graphs. *Discrete Applied Mathematics*, *70*(3), 217–229. https://doi.org/10.1016/0166-218X(95)00106-2.

[31] A. N. A. Koam, A. Ahmad, M. S. Alatawi, M. Azeem & M. F. Nadeem (2023). Metric basis of four-dimensional Klein bottle. *CMES-Computer Modeling in Engineering and Sciences*, *136*(3), 3011–3024. https://doi.org/10.32604/cmes.2023.024764.

[32] J. Kratica, V. Kovačević-Vujčić, M. Čangalović & M. Stojanović (2012). Minimal doubly resolving sets and the strong metric dimension of some convex polytopes. *Applied Mathematics and Computation*, *218*(19), 9790–9801. https://doi.org/10.1016/j.amc.2012.03.047.

[33] K. Liu & N. Abu-Ghazaleh (2006). Virtual coordinates with backtracking for void traversal in geographic routing. In *International Conference on Ad-Hoc Networks and Wireless*, volume 4104 pp. 46–59. Berlin, Heidelberg. Springer. https://doi.org/10.1007/11814764_6.

[34] D. M. Mahanani & D. Ismiarti (2024). A bipartite graph associated to elements and class equivalences of a finite heap. *Malaysian Journal of Mathematical Sciences*, *18*(4), 807–819. https://doi.org/10.47836/mjms.18.4.08.

[35] P. Manuel, R. Bharati & I. Rajasingh (2008). On minimum metric dimension of honeycomb networks. *Journal of Discrete Algorithms*, *6*(1), 20–27. https://doi.org/10.1016/j.jda.2006.09.002.

[36] P. Manuel, B. Rajan, I. Rajasingh & M. C. Monica (2006). Landmarks in torus networks. *Journal of Discrete Mathematical Sciences and Cryptography*, *9*(2), 263–271. https://doi.org/10.1080/09720529.2006.10698077.

[37] P. D. Manuel, M. I. Abd-El-Barr, I. Rajasingh & B. Rajan (2008). An efficient representation of Benes networks and its applications. *Journal of Discrete Algorithms*, *6*(1), 11–19. https://doi.org/10.1016/j.jda.2006.08.003.

[38] I. Masmali, M. T. A. Kanwal, M. K. Jamil, A. Ahmad, M. Azeem & A. N. A. Koam (2024). COVID antiviral drug structures and their edge metric dimension. *Molecular Physics*, *122*(5), Article ID: e2259508. https://doi.org/10.1080/00268976.2023.2259508.

[39] R. A. Melter & I. Tomescu (1984). Metric bases in digital geometry. *Computer Vision, Graphics, and Image Processing*, *25*(1), 113–121. https://doi.org/10.1016/0734-189X(84)90051-3.

[40] M. F. Nadeem, A. Shabbir & M. Azeem (2024). On metric dimension of carbon nanotube Y-junctions. *Journal of Applied Mathematics and Computing*, *70*, 6013–6034. https://doi.org/10.1007/s12190-024-02208-y.

[41] R. Nawaz, M. K. Jamil & M. Azeem (2024). Edge-based metric resolvability of anti-depression molecular structures and its application. *Results in Chemistry*, *7*, Article ID: 101458. https://doi.org/10.1016/j.rechem.2024.101458.

[42] C. Poisson & P. Zhang (2002). The metric dimension of unicyclic graphs. *Journal of Combinatorial Mathematics and Combinatorial Computing*, *40*, 17–32.

[43] S. Prabhu, D. S. R. Jeba & S. Stephen (2025). Metric dimension of star fan graph. *Scientific Reports*, *15*(1), Article ID: 102. https://doi.org/10.1038/s41598-024-83562-6.

[44] S. Prabhu, V. Manimozhi, A. Davoodi & J. L. G. Guirao (2024). Fault-tolerant basis of generalized fat trees and perfect binary tree derived architectures. *The Journal of Supercomputing*, *80*(11), 15783–15798. https://doi.org/10.1007/s11227-024-06053-5.

[45] S. Prabhu, V. Manimozhi, M. Arulperumjothi & S. Klavžar (2022). Twin vertices in fault-tolerant metric sets and fault-tolerant metric dimension of multistage interconnection networks. *Applied Mathematics and Computation*, *420*, Article ID: 126897. https://doi.org/10.1016/j.amc.2021.126897.

[46] S. Prabhu, D. Sagaya Rani Jeba, M. Arulperumjothi & S. Klavžar (2024). Metric dimension of irregular convex triangular networks. *AKCE International Journal of Graphs and Combinatorics*, *21*(3), 225–231. https://doi.org/10.1080/09728600.2023.2280799.

[47] B. Rajan, I. Rajasingh, J. A. Cynthia & P. Manuel (2014). Metric dimension of directed graphs. *International Journal of Computer Mathematics*, *91*(7), 1397–1406. https://doi.org/10.1080/00207160.2013.844335.

[48] B. Rajan, I. Rajasingh, M. C. Monica & P. Manuel (2008). Metric dimension of enhanced hypercube networks. *Journal of Combinatorial Mathematics and Combinatorial Computing*, *67*, 5–15.

[49] B. Rajan, I. Rajasingh, P. Venugopal & M. C. Monica (2014). Minimum metric dimension of Illiac networks. *Ars Combinatoria*, *117*, 95–103.

[50] B. Rajan, K. T. Sonia & M. C. Monica (2011). Conditional resolvability of honeycomb and hexagonal networks. *Mathematics in Computer Science*, *5*(1), 89–99. https://doi.org/10.1007/s11786-011-0076-3.

[51] A. Sebő & E. Tannier (2004). On metric generators of graphs. *Mathematics of Operations Research*, *29*(2), 383–393. https://doi.org/10.1287/moor.1030.0070.

[52] Z. Shao, S. M. Sheikholeslami, P. Wu & J. B. Liu (2018). The metric dimension of some generalized Petersen graphs. *Discrete Dynamics in Nature and Society*, *2018*(1), Article ID: 4531958. https://doi.org/10.1155/2018/4531958.

[53] H. M. A. Siddiqui & M. Imran (2014). Computing the metric dimension of wheel related graphs. *Applied Mathematics and Computation*, *242*, 624–632. https://doi.org/10.1016/j.amc.2014.06.006.

[54] P. Slater (1975). Leaves of trees. *Congressus Numerantium*, *14*, 549–559.

[55] S. Söderberg & H. S. Shapiro (1963). A combinatory detection problem. *The American Mathematical Monthly*, *70*(10), 1066–1070. https://doi.org/10.1080/00029890.1963.11992174.

[56] R. C. Tillquist, R. M. Frongillo & M. E. Lladser (2023). Getting the lay of the land in discrete space: A survey of metric dimension and its applications. *SIAM Review*, *65*(4), 919–962. https://doi.org/10.1137/21M1409512.

[57] T. Vetrík (2017). The metric dimension of circulant graphs. *Canadian Mathematical Bulletin*, *60*(1), 206–216. https://doi.org/10.4153/CMB-2016-048-1.

[58] T. Xie, C. He, X. Ren, C. Shahabi & C. C. J. Kuo (2022). L-BGNN: Layerwise trained bipartite graph neural networks. *IEEE Transactions on Neural Networks and Learning Systems*, *34*(12), 10711–10723. https://doi.org/10.1109/TNNLS.2022.3171199.

[59] Y. Zhang, C. A. Phillips, G. L. Rogers, E. J. Baker, E. J. Chesler & M. A. Langston (2014). On finding bicliques in bipartite graphs: A novel algorithm and its application to the integration of diverse biological data types. *BMC Bioinformatics*, *15*, 1–18. https://doi.org/10.1186/1471-2105-15-110.